UNIMODALITY AND EVENTUAL LOG-CONCAVITY OF SUBSUM POLYNOMIALS

Abstract. We prove that the binary-partition numerator of Ballantine, Beck, Feigon and Maurischat is unimodal for every $n \geq 2$, resolving their revised Conjecture 6. We also prove their denominator log-concavity conjecture for every $n \geq 2^{10^8}$, with strict inequalities at all internal coefficients. The unimodality argument combines an exact binary recurrence with a signed residual and a quantitative central-slope invariant. The denominator argument combines coefficient estimates at the edges with variance-relative Taylor bounds and a Fourier positivity criterion. Both results are proved in Lean 4 for the original partition polynomials. Full denominator log-concavity and revised Conjecture 7 on binary log-concavity remain open.

1. Introduction

1.1. **Background and main results.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, its subsum polynomial is

$$\mathrm{sp}(\lambda, x) = \prod_{j=1}^{\ell} (1 + x^{\lambda_j}).$$

Ballantine, Beck, Feigon and Maurischat [1] study sums of its reciprocal over ordinary and restricted partitions. Their conjectures ask, among other things, whether the coefficients of the resulting numerators and denominators have prescribed shapes.

Write $N_n(x) = \mathrm{num}_{\mathcal{B}}(n, x)$ for the numerator associated with binary partitions, whose parts are powers of two. Write $D_n(x) = \mathrm{den}(n, x)$ for the ordinary-partition denominator after the common polynomial factor has been removed. These are the specific normalizations defined in §2. A polynomial is *unimodal* if its coefficients increase weakly to a peak and then decrease weakly. A nonnegative coefficient sequence (a_k) is *log-concave* if $a_{k-1}a_{k+1} \leq a_k^2$ at every internal index.

Theorem 1.1 (Revised BBFM Conjecture 6)**.** *For every integer $n \geq 2$, the polynomial $N_n(x)$ is unimodal.*

The proof gives the peak explicitly and includes the zero coefficients beyond the degree. Our second result establishes an explicit eventual form of BBFM's Conjecture 4.

Theorem 1.2 (Eventual denominator log-concavity)**.** *Let $n \geq 2^{10^8}$ and write $D_n(x) = \sum_{k=0}^{d_n} a_{n,k} x^k$. Then*

$$a_{n,k-1}a_{n,k+1} < a_{n,k}^2 \qquad (1 \leq k < d_n). \tag{1.1}$$

In particular, $D_n(x)$ is log-concave.

The full Conjecture 4 asks for log-concavity at every positive n outside $\{3, 5, 6, 7\}$. Theorem 1.2 leaves an explicit, very large finite range unresolved. The constant is a sufficient bound obtained from the estimates below; no optimality is asserted.

Both theorems have complete Lean 4 proofs in the accompanying `bbfm-conjectures` repository [4]. The proof statements refer to the partition-sum numerator and the common-gcd denominator, respectively. Their mathematical dependencies, including the analytic estimates and finite certificates, are proved within the formal development.

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1.2. **Relation to earlier work.** We use the numbering in the revised BBFM paper [1]: Conjecture 6 is binary unimodality, and Conjecture 7 is binary log-concavity for $n \geq 6$. Chen, Ono and Zhang [2, §1.2.2] display Conjecture 6 as unimodality for $n > 5$, and discuss the original log-concavity claim in the accompanying remark. Their Table 1 groups the two properties and gives the status "Disproved (corrected $n > 5$ form open)". The counterexample at $n = 4$ has coefficients

$$(4, 10, 18, 18, 20, 18, 18, 10, 4). \tag{1.2}$$

The inequality $18^2 < 18 \cdot 20$ disproves log-concavity, whereas this sequence is unimodal. Their discussion explicitly observes that "The coefficient sequence is unimodal, but it is not log-concave" [2, §1.2.2]. Thus their counterexample leaves the unimodality assertion untouched. Theorem 1.1 proves that assertion for the complete range $n \geq 2$.

Their Table 1 also records Conjecture 4 as open. Theorem 1.2 advances that question by proving all coefficient inequalities above an explicit threshold. It does not replace a numerical bound previously claimed by Chen, Ono and Zhang. The stronger binary log-concavity assertion remains a separate open problem; §5 records the partial results obtained here.

1.3. **Outline of the proofs.** The exact binary recurrence expresses N_{2t} as the sum of a geometric multiple of N_{2t-2} and a binomial smoothing of $N_t(x^2)$. The geometric factor has gaps in its coefficients, so a direct appeal to preservation of unimodality under products does not give an induction. We instead remove a signed central monomial and divide by $1 + x$. The resulting polynomial has odd symmetric degree, which makes the geometric multiplication compatible with unimodality. Its recurrence contains a signed correction. A lower bound of 3^n for the source slopes in a central window pays for that correction at the next step. The decisive estimate comes from subtracting a centered triangle and following its slopes through binomial smoothing. Exact certificates settle the initial even sizes through 96.

For the denominator, a product formula of BBFM identifies D_n with a product of weighted Bernoulli factors. Exponential tilting reduces a coefficient inequality to positivity of a centered Fourier integral. The main quantitative refinement bounds the cubic logarithmic remainder by the variance times an effective length scale. A Gaussian estimate then controls the possible negative contribution near the origin, while a separate decay estimate controls the rest of the integral. An elementary coefficient-ratio argument treats the edge indices where the analytic parameter is small. Reflection completes the upper half.

Sections 3 and 4 give these arguments, with explicit analytic constants. Appendix A identifies the principal Lean declarations and their source files.

2. The partition polynomials

2.1. **Binary numerator.** Let $\mathcal{B}(n)$ be the set of partitions of n into powers of two, and let $m_\lambda(i)$ be the multiplicity of i in λ. The polynomial used throughout the binary argument is

$$N_n(x) = \sum_{\lambda \in \mathcal{B}(n)} \prod_{j \geq 0} (1 + x^{2^j})^{\lfloor n/2^j \rfloor - m_\lambda(2^j)}. \tag{2.1}$$

Only finitely many factors have nonzero exponents. This is the literal definition of `numB` in the formalization; the recurrence used below is a theorem about (2.1).

Set

$$c_n = \sum_{j \geq 0} 2^j \left\lfloor \frac{n}{2^{j+1}} \right\rfloor. \tag{2.2}$$

Every summand in (2.1) has nonnegative coefficients and is symmetric of degree $2c_n$. Indeed, its degree is $\sum_{j \geq 0} 2^j \lfloor n/2^j \rfloor - n = 2c_n$. Consequently, N_n is nonnegative and symmetric about c_n. We write $P[k] = [x^k]P$ and extend coefficients by zero to all integers outside the support. Put $\Delta P[k] = P[k] - P[k-1]$.

2.2. **Ordinary denominator.** For ordinary partitions of n, put

$$H_\lambda(x) = \prod_{i=1}^{n} (1+x^i)^{\lfloor n/i \rfloor - m_\lambda(i)}, \qquad D_n^*(x) = \prod_{i=1}^{n} (1+x^i)^{\lfloor n/i \rfloor},$$
$$G_n(x) = \gcd_{\lambda \vdash n} H_\lambda(x), \qquad D_n(x) = D_n^*(x)/G_n(x), \tag{2.3}$$

where the gcd is monic over $\mathbb{Q}[x]$. The distinction between this common gcd and a gcd chosen only after summing the H_λ is part of the definition.

The product formula of BBFM [1, Proposition 3.8] is

$$D_n(x) = \prod_{i=1}^{n} (1+x^i)^{m_{n,i}}, \qquad m_{n,i} = 1 + \left\lfloor \log_2 \frac{n}{i} \right\rfloor. \tag{2.4}$$

For $1 \le i \le n$, the multiplicities are positive, nonincreasing, and bounded above by

$$s_n = m_{n,1} = 1 + \lfloor \log_2 n \rfloor. \tag{2.5}$$

The product is monic, nonnegative and symmetric. Its degree is $d_n = \sum_{i=1}^{n} i\, m_{n,i}$.

The formal source bridge proves (2.4) for (2.3). It compares complex root multiplicities of the two monic polynomials and then uses their complete splitting over $\mathbb{C}$. The root-multiplicity input includes a credited excerpt from Axiom's Conjecture 2 development [3]. Mapping coefficients to $\mathbb{R}$ then permits the analytic argument; the final inequalities are transported back to $\mathbb{Q}$.

3. Unimodality of the binary numerator

3.1. **The recurrence and the residual.** Write $U_q(x) = 1 + x + \cdots + x^{q-1}$. The binary partition bijections give

$$N_{2m+1}(x) = N_{2m}(x), \tag{3.1}$$
$$N_{2t}(x) = U_q(x^2) N_{2t-2}(x) + (1+x)^{2t} N_t(x^2), \tag{3.2}$$

where $t \ge 1$ and $q = 2^{v_2(t)+1} \le 2t$. For (3.1), append a part of size one. For (3.2), split the binary partitions of $2t$ according to whether they contain a part of size one. In the first class, remove two ones; in the second, divide all parts by two. The changes in the floor exponents in (2.1) give the two factors displayed above. The centers satisfy

$$c_{2m+1} = c_{2m}, \qquad c_{2t} = t + 2c_t, \qquad c_{2t} = c_{2t-2} + q - 1. \tag{3.3}$$

Define

$$A_n = N_n(-1), \qquad R_n(x) = \frac{N_n(x) - (-1)^{c_n} A_n x^{c_n}}{1+x}. \tag{3.4}$$

The numerator vanishes at -1, so this is a polynomial in $\mathbb{Z}[x]$. For $c_n > 0$, it is symmetric with degree bound $2c_n - 1$. Evaluation of (3.2) at -1 gives

$$A_{2t} = q A_{2t-2}, \qquad A_{2t} = 2^{t + v_2(t!)}, \qquad 0 < A_n \le 2^n. \tag{3.5}$$

The factorial formula follows by induction, and the last inequality uses $v_2(t!) \le t$ and (3.1).

Let Q_q be the polynomial specified by

$$(1+x) Q_q(x) = U_q(x^2) + q x^{q-1}. \tag{3.6}$$

Here q is even, so the right side vanishes at -1. Substituting (3.4) into (3.2) gives

$$R_{2t}(x) = U_q(x^2) R_{2t-2}(x) + (1+x)^{2t-1} N_t(x^2) + (-1)^{c_{2t-2}} A_{2t-2} x^{c_{2t-2}} Q_q(x). \tag{3.7}$$

The coefficients of Q_q lie between $-q$ and q. One checks this below its center from (3.6), and uses symmetry for the other half. Thus, if E_t denotes the last summand in (3.7),

$$|E_t[k]| \le A_{2t}, \qquad |\Delta E_t[k]| \le 2A_{2t} \le 2 \cdot 2^{2t}. \tag{3.8}$$

Moreover, $E_t[k] = 0$ for $k < c_{2t-2}$.

3.2. **Shape and quantitative smoothing.** Say that P has *shape* d if its coefficients are nonnegative, vanish above d, satisfy $P[k] = P[d-k]$ for $0 \le k \le d$, and are weakly increasing up to the center. The following closure facts are used in the induction.

Lemma 3.1. *If P has shape d, then $(1+x)^r P(x^2)$ has shape $2d+r$ for every $r \ge 1$. If P has odd shape $2c+1$, then $U_q(x^2)P(x)$ has shape $2c+2q-1$ for every $q \ge 1$.*

Proof. Multiplication of $P(x^2)$ by $1+x$ repeats each coefficient twice; further multiplication by $1+x$ preserves symmetry and monotonicity on the lower half. For the second assertion, extend the coefficient sequence by zero to $\mathbb{Z}$. The coefficients of the product are $\sum_{j=0}^{q-1} P[k-2j]$. Pair terms across the half-integer center of P in the difference of consecutive such sums. Reflection cancels the terms on opposite sides, and the remaining differences lie on the increasing half. This proves lower-half monotonicity. Support, nonnegativity and reflection follow directly from the product. □

The odd shape in the second assertion is the reason for introducing R_n. The recurrence for this residual has a geometric term whose first differences are already nonnegative. The remaining issue is to make the smoothing term dominate (3.8).

For $W < c$, define the centered triangle

$$T_{c,W}(x) = x^{c-W} U_{W+1}(x)^2. \tag{3.9}$$

Its first difference is 1 on $c - W \le k \le c$ and 0 below that interval. If P has shape $2c$ and $\Delta P[k] \ge K \ge 0$ on this interval, then $P - KT_{c,W}$ still has shape $2c$. This follows by subtracting first differences on the lower half, observing that the constant coefficient is unchanged, and reflecting.

Lemma 3.2 (Central-slope transfer)**.** *Suppose $t \ge 3$, $t+1 \le W < c$, and P has shape $2c$ with $\Delta P[k] \ge K \ge 0$ for $c - W \le k \le c$. If $C = t + 2c$, then*

$$\Delta\big((1+x)^{2t-1}P(x^2)\big)[k] \ge K\binom{2t-3}{t-2} \quad (C-(2t+2) \le k < C). \tag{3.10}$$

Proof. Subtract $KT_{c,W}$ and apply Lemma 3.1 to the remainder. Its smoothed first differences are nonnegative in the indicated range. For the triangle itself, the identity $(1+x)U_{W+1}(x^2) = U_{2W+2}(x)$ gives

$$(1+x)^{2t-1}T_{c,W}(x^2) = x^{2(c-W)}(1+x)^{2t-3}U_{2W+2}(x)^2. \tag{3.11}$$

Set $s = t-2$, $J = 2W+2$ and $z = k - 2(c-W)$. The hypotheses give $s \le z \le J+s-1$ and $J \ge 2s+2$. Taking a first difference telescopes one of the uniform factors in (3.11). Writing $B_j = \binom{2s+1}{j}$, the resulting difference is

$$\begin{cases} \displaystyle\sum_{j=0}^{z} B_j, & z < J, \\ \displaystyle 2^{2s+1} - 2\sum_{j=0}^{z-J} B_j, & z \ge J. \end{cases}$$

The first sum contains B_s. In the second case $z - J < s$, and symmetry gives $\sum_{j=0}^{s} B_j = 2^{2s}$, so the expression is at least $2B_s$. Adding back the nonnegative remainder proves (3.10). □

We also use the explicit growth estimate

$$3^{2t} + 4 \cdot 2^{2t} \le 3^{t-1}\binom{2t-3}{t-2} \qquad (t \ge 48). \tag{3.12}$$

To see the source of the bound, the largest coefficient of $(1+x)^{2t-3}$ is at least $2^{2t-3}/(2t-2)$. It therefore suffices to check

$$8(2t-2)(3^{2t} + 4 \cdot 2^{2t}) \le 3^{t-1}2^{2t}.$$

The base case $t = 48$ is an exact integer comparison. Thereafter the right side grows by a factor of 12, which dominates the growth of each term on the left. The formal proof performs this induction and the base comparison in exact arithmetic.

3.3. **The simultaneous invariant.** For even $n \ge 48$, let $\mathcal{S}(n)$ be the conjunction of

$$\begin{aligned} &N_n \text{ has shape } 2c_n, \\ &R_n \text{ has shape } 2c_n - 1, \\ &\Delta N_n[k] \ge 3^n \quad (c_n - (n+2) \le k \le c_n). \end{aligned} \tag{3.13}$$

The source and residual have different centers; the slope assertion is a property of the source polynomial.

Proposition 3.3 (Induction step)**.** *For $t \ge 49$,*

$$\mathcal{S}(2t-2) \text{ and } \mathcal{S}(2\lfloor t/2 \rfloor) \quad \Longrightarrow \quad \mathcal{S}(2t).$$

Proof. Put $H = 2\lfloor t/2 \rfloor$. Then $N_t = N_H$ and $c_t = c_H$ by (3.1). Apply Lemma 3.2 with $P = N_H$, $W = H+2$ and $K = 3^H$. The center bounds used here give $W < c_H$; also $W \ge t+1$ and $H \ge t-1$. By (3.12), the smoothing term in (3.7) has first difference at least $3^{2t} + 4 \cdot 2^{2t}$ on the required central window.

The geometric term has nonnegative first differences below the new residual center by Lemma 3.1. Subtracting the possible loss in (3.8) leaves

$$\Delta R_{2t}[k] \ge 3^{2t} + 2 \cdot 2^{2t} \quad (c_{2t} - (2t+2) \le k < c_{2t}). \tag{3.14}$$

Below this window the correction vanishes: use $c_{2t} = c_{2t-2} + q - 1$ and $q \le 2t$. Both remaining terms are increasing there. The residual is therefore increasing on its full lower half. Its constant coefficient is nonnegative, and its known reflection and support give its complete shape.

Finally reconstruct the source:

$$N_{2t}(x) = (1+x)R_{2t}(x) + (-1)^{c_{2t}} A_{2t} x^{c_{2t}}. \tag{3.15}$$

Away from the central coefficient, multiplication by $1 + x$ preserves the lower-half inequalities. At the last step into the center, reflection of R_{2t} gives the exact identity

$$\Delta N_{2t}[c_{2t}] = \Delta R_{2t}[c_{2t} - 1] + (-1)^{c_{2t}} A_{2t}.$$

Equations (3.14) and (3.5) make this at least 3^{2t}. The other central slopes satisfy the same bound by (3.15). This proves all parts of $\mathcal{S}(2t)$. □

3.4. **Finite certificates and completion.** The initial data consist of the exact coefficient lists for every even size $2 \le n \le 96$. They prove source shape throughout that range, and all three parts of (3.13) for $48 \le n \le 96$. The certificate mechanism identifies each proposed list with the partition polynomial before using its inequalities.

Here is the underlying arithmetic observation. If $P \in \mathbb{Z}[x]$ has nonnegative coefficients and $P(1) < b$, then every coefficient is less than b. Thus $P(b)$ is a base-b encoding of its coefficient

list with no carries. A proposed nonnegative list $(u_0, \dots, u_d)$, whose entries are all below b, is the coefficient list of P if

$$P(b) = \sum_{j=0}^{d} u_j b^j. \tag{3.16}$$

Uniqueness of base expansion proves the assertion. The source recurrence supplies the evaluation identities, and exact list checks supply symmetry, monotonicity and the central-slope bounds. Residual lists are identified by the polynomial identity (3.4). The finite arithmetic is checked by Lean's kernel.

Proof of Theorem 1.1. Strong induction on t proves $\mathcal{S}(2t)$ for all $t \geq 24$. The cases $24 \leq t \leq 48$ are certified. For $t \geq 49$, both $t-1$ and $\lfloor t/2 \rfloor$ are smaller indices at least 24, so Proposition 3.3 applies. The smaller even sizes are already covered by the source certificates. Equation (3.1) covers every odd $n \geq 3$. In all cases the peak is c_n, and reflection gives the decreasing half. Nonnegativity and the support bound include the zero tail in the unimodality statement. □

4. Eventual log-concavity of the denominator

4.1. **Tilting and a Fourier criterion.** We first work with the real polynomial on the right of (2.4). More generally, for a finite multiplicity profile $m_1, \dots, m_n$, set

$$P(x) = \prod_{i=1}^{n} (1 + x^i)^{m_i} = \sum_j a_j x^j.$$

At a positive radius r, the normalized numbers $a_j r^j / P(r)$ form a probability distribution. Its mean and variance are

$$\mu(r) = \sum_{i=1}^{n} \frac{m_i i r^i}{1 + r^i}, \qquad V(r) = \sum_{i=1}^{n} \frac{m_i i^2 r^i}{(1 + r^i)^2}. \tag{4.1}$$

These formulas also follow directly by logarithmic differentiation. For the denominator profile, μ increases continuously from 0 to $d_n/2$ as r increases from 0 to 1. In fact, $r\mu'(r) = V(r) > 0$. Therefore, for every lower-half index $1 \leq k \leq d_n/2$, there is a radius $0 < r \leq 1$ with $\mu(r) = k$.

Define the centered characteristic function

$$\phi_{n,k,r}(\theta) = \frac{D_n(re^{i\theta})}{D_n(r)} e^{-ik\theta}. \tag{4.2}$$

Its real part is even. Orthogonality of the exponential functions gives, with $b_j = a_{n,j} r^j$,

$$\int_{-\pi}^{\pi} \operatorname{Re} \phi_{n,k,r}(\theta)(1 - \cos\theta)\, d\theta = \frac{\pi}{D_n(r)}(2b_k - b_{k-1} - b_{k+1}). \tag{4.3}$$

If the integral is positive, then $2b_k > b_{k-1} + b_{k+1}$. By nonnegativity and the arithmetic–geometric mean inequality, $b_k^2 > b_{k-1} b_{k+1}$. Cancelling r^{2k} proves the desired strict log-concavity inequality. The choice $\mu(r) = k$ centers the local expansion in (4.2).

4.2. **The variance-relative moment bound.** Put

$$M_3(r) = \sum_{i=1}^{n} m_i i^3 r^i, \qquad L = L_n(r) = \begin{cases} \min\{n, (-\log r)^{-1}\}, & r < 1, \\ n, & r = 1. \end{cases}$$

The useful estimate compares the raw third moment directly with the variance. It holds for every nonincreasing nonnegative integer profile.

Lemma 4.1. *Suppose $n \geq 1$ and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. For $1/2 < r \leq 1$,*

$$M_3(r) \leq \frac{9}{2} L_n(r) V(r). \tag{4.4}$$

For $0 \leq r \leq 1/2$,

$$M_3(r) \leq 6V(r). \tag{4.5}$$

Proof. The passage from constant weights on prefixes to decreasing weights is summation by parts. If $A_q = \sum_{i=1}^{q} u_i$ and $A_q \leq 0$ for every $q \leq n$, then

$$\sum_{i=1}^{n} m_i u_i = m_n A_n + \sum_{q=1}^{n-1} (m_q - m_{q+1}) A_q \leq 0. \tag{4.6}$$

It suffices, therefore, to prove the two moment comparisons for each finite prefix with constant multiplicity one.

For $1/2 < r < 1$, write $h = -\log r$. The infinite constant-profile comparison is

$$\sum_{i \geq 1} i^3 r^i \leq \frac{4}{h} \sum_{i \geq 1} \frac{i^2 r^i}{(1 + r^i)^2}.$$

To prove it, use $q/(1+q)^2 \geq q - 2q^2$ to bound the variance below, substitute the exact geometric second and third moments, and clear the positive denominators. A rational upper bound for $-\log r$, verified by differentiation, completes the comparison. All series involved are absolutely summable.

For a prefix reaching at least $4/h$, the removed tail has $i^3 r^i \geq (4/h) i^2 r^i / (1 + r^i)^2$, so subtracting it preserves the comparison. For a shorter prefix, the elementary inequality $x(1 + e^{-x})^2 \leq 9/2$ on $0 \leq x \leq 4$ gives a termwise bound. If the effective scale is instead saturated at $L = n$, the simpler inequality $i(1 + r^i)^2 \leq 4n$ suffices. These arguments give (4.4) for every prefix, and (4.6) handles the weights.

For $r \leq 1/2$, the same geometric-series calculation yields the infinite comparison with constant 6. Prefixes ending at an index at most 5 are checked by the inequalities $i(1 + r^i)^2 \leq 6$ for those indices. Longer prefixes follow by subtracting a tail whose terms have ratio at least 6. Apply (4.6) again. The comparison therefore holds at every finite length. □

The finite-prefix statements and the infinite comparisons are proved in the `Bernoulli` moment modules. In the formal statements the profile is a function on $\mathbb{N}$; it may be extended by zero after its last nonzero value.

The logarithmic derivative of an individual Bernoulli factor admits a uniform third-derivative bound on the local integration interval. Summing those bounds contributes a factor $4/3$ to M_3; Taylor's theorem then contributes $1/6$. At the saddle radius, this proves

$$\begin{aligned} &\phi_{n,k,r}(\theta) = \exp\{-V\theta^2/2 + R(\theta)\}, && \\ &|R(\theta)| \leq VL\theta^3 && \left(r > \tfrac{1}{2},\ 0 \leq \theta \leq \frac{1}{4L}\right), \\ &|R(\theta)| \leq \tfrac{4}{3} V\theta^3 && \left(r \leq \tfrac{1}{2},\ 0 \leq \theta \leq \tfrac{1}{4}\right). \end{aligned} \tag{4.7}$$

The formal argument constructs the local logarithmic expansion and proves the exponential identity in (4.7); no choice of logarithm or unproved analytic approximation is supplied as an assumption to the source theorem.

4.3. **Quantitative Fourier positivity.** The following criterion records the constants used to turn (4.7) into a strict coefficient inequality.

Proposition 4.2. *Let $\phi : \mathbb{R} \to \mathbb{C}$ be continuous with even real part. Suppose N, L, V satisfy*

$$N \geq 12 \cdot 10^6, \qquad 1 \leq L \leq N, \qquad \frac{NL^2}{100} \leq V \leq 100NL^2. \tag{4.8}$$

Assume, for $0 \leq \theta \leq \pi$,

$$|\phi(\theta)| \leq \exp\left\{-\frac{N}{5000}\min(1, L^2\theta^2)\right\}, \tag{4.9}$$

and, for $0 \leq \theta \leq 1/(4L)$,

$$\phi(\theta) = \exp\{-V\theta^2/2 + R(\theta)\}, \qquad |R(\theta)| \leq VL\theta^3.$$

Then

$$\int_{-\pi}^{\pi} \operatorname{Re}\phi(\theta)(1 - \cos\theta)\, d\theta > 0. \tag{4.10}$$

Proof. Set $a = V^{-1/2}$, $c = (4L)^{-1}$ and $M = VL$. The parameter bounds give $0 < a \leq c \leq 1$ and

$$M^2a^6 = \frac{L^2}{V} \leq 10^{-5}. \tag{4.11}$$

On $[0, a]$, the local expansion implies $\operatorname{Re}\phi \geq 1/4$. The inequality $1 - \cos\theta \geq 9\theta^2/20$ for $0 \leq \theta \leq 1$ therefore gives

$$\int_0^a \operatorname{Re}\phi(\theta)(1 - \cos\theta)\, d\theta \geq \frac{3a^3}{80}.$$

On $[0, c]$, the cubic remainder is at most $V\theta^2/4$. Bounding the possible negative real part of its exponential by the square of its imaginary part yields

$$\operatorname{Re}\phi(\theta) \geq -M^2\theta^6 e^{-V\theta^2/4}.$$

Using $1 - \cos\theta \leq \theta^2/2$ and the Gaussian moment bound

$$\int_0^c \theta^8 e^{-(\theta/a)^2/4}\, d\theta \leq 6000a^9,$$

we obtain, by (4.11),

$$\int_a^c \operatorname{Re}\phi(\theta)(1 - \cos\theta)\, d\theta \geq -\frac{3a^3}{100}.$$

On $[c, \pi]$, equation (4.9) gives $|\phi(\theta)| \leq \delta = e^{-N/80000}$. The explicit exponential estimate used in the formalization is

$$e^{-N/80000} \leq \frac{a^3}{10000}.$$

Here $V \leq 100N^3$ follows from (4.8); this upper bound and $N \geq 12 \cdot 10^6$ suffice for the exponential comparison. Since $1 - \cos\theta \leq 2$ and $\pi < 4$, the far contribution is at least -8δ. Adding the three ranges gives a positive lower bound, since

$$\frac{3}{80} - \frac{3}{100} - \frac{8}{10000} > 0.$$

Evenness doubles the positive half-integral and proves (4.10). □

The improvement in the local estimate is visible in (4.11): the squared normalized error is L^2/V. The variance itself controls the cubic remainder. There is no need to deduce that control from a separate coarse lower bound for the variance at every occurrence of a third moment.

4.4. **Discharging the analytic hypotheses.** For the denominator product, the needed bounds hold with explicit parameters. We record them to explain the final threshold.

When $0 < r \le 1/2$, set $Q = s_n r$. Direct estimates in (4.1) give

$$\mu(r) \le 4Q, \qquad Q/4 \le V(r) \le 16Q. \tag{4.12}$$

The unit-weight factors give

$$|\phi_{n,k,r}(\theta)| \le e^{-Q\min(1,\theta^2)/200} \qquad (|\theta| \le \pi).$$

Together with (4.7), these estimates discharge Proposition 4.2 with

$$N = 14Q, \qquad L = 4/3, \tag{4.13}$$

provided $14Q \ge 12 \cdot 10^6$. The constants in this rescaling are compatible with both the variance lower bound and the Fourier decay.

When $1/2 < r \le 1$, take $L = L_n(r)$ and set

$$N = L\big(1 + \log_2(n/L)\big). \tag{4.14}$$

Then $1 \le L \le n$, and the formal source estimates give

$$\frac{NL^2}{100} \le V(r) \le 20NL^2, \qquad |\phi_{n,k,r}(\theta)| \le e^{-N\min(1,L^2\theta^2)/5000}. \tag{4.15}$$

The decay estimate comes from a consecutive block of factor indices at the effective scale. The squared sine terms in the modulus of each Bernoulli factor control its loss from one; summing those terms over the block supplies the minimum in (4.15). The dyadic multiplicities provide the factor $1+\log_2(n/L)$. The variance lower bound uses the same scale. These estimates, and the large-radius expansion in (4.7), apply to the concrete product without additional hypotheses. Thus $N \ge 12\cdot 10^6$ again suffices.

4.5. **The edge estimate.** The analytic parameters can be small close to an endpoint. The following algebraic estimate covers that range.

Lemma 4.3. *Let $P(x) = \prod_{i=1}^{n}(1+x^i)^{m_i} = \sum_j a_j x^j$, where $n \ge 1$, $0 \le m_i \le m_1$, and $k \ge 1$. If $28k \le m_1$, then $a_{k-1}a_{k+1} < a_k^2$.*

Proof. Write $s = m_1$ and introduce the normalized ratios $\rho_j = (j+1)a_{j+1}/a_j$. The factor $(1+x)^s$ gives positivity through the required indices and the lower bound $\rho_j \ge s - j$. Logarithmic differentiation gives a finite coefficient recurrence

$$(j+1)a_{j+1} = \sum_{t=0}^{j} b_t a_{j-t}, \qquad b_0 = s, \tag{4.16}$$

where the b_t are the coefficients of P'/P as a formal power series. Bounding the other multiplicities by s gives the rough estimates $s - j \le \rho_j \le s + 8j$ on the initial range. Substitution of these estimates back into (4.16) gives

$$|\rho_k - \rho_{k-1}| \le 16 \qquad (28k \le s).$$

These finite ratio bounds are proved in the `NewtonTwentyEight` development; no previous log-concavity inequality is assumed. Since $\rho_{k-1} \ge s - k > 16k$, we have $k\rho_k < (k+1)\rho_{k-1}$. Expanding the ratios and cancelling positive factors gives $a_{k-1}a_{k+1} < a_k^2$. □

4.6. **Completion of the eventual theorem.**

Proof of Theorem 1.2. We prove the stronger sufficient condition $s_n \geq 10^8$. First suppose $1 \leq k \leq d_n/2$. If $28k \leq s_n$, apply Lemma 4.3 to (2.4). Otherwise choose the saddle radius r with $\mu(r) = k$.

If $r \leq 1/2$, equations (4.12) and (4.13) give

$$N = 14 s_n r \geq \frac{7}{2} k > \frac{s_n}{8} \geq 12.5 \cdot 10^6 > 12 \cdot 10^6.$$

If $r > 1/2$, the effective-scale comparison gives

$$N = L(1 + \log_2(n/L)) \geq s_n/2 \geq 50 \cdot 10^6.$$

In either case Proposition 4.2, with the discharged source estimates, proves positivity in (4.3) and hence the strict coefficient inequality.

For $d_n/2 < k < d_n$, use the already proved inequality at $d_n - k$ and the symmetry $a_{n,j} = a_{n,d_n-j}$. Finally, $n \geq 2^{10^8}$ implies $s_n \geq 10^8$, and the exact source bridge transports the result to (2.3). At the degree and beyond it the non-strict inequalities follow from the zero tail. □

5. Further coefficient ranges and open questions

5.1. **Uniform denominator interior.** The same analytic criterion gives a result independent of the size of n.

Theorem 5.1. *Let $W_0 = 6(12 \cdot 10^6)^2 = 864000000000000$. For every positive n, the inequality in (1.1) holds whenever $k \geq W_0$ and $d_n - k \geq W_0$.*

Proof. By symmetry it is enough to consider $k \leq d_n/2$. At a small-radius saddle, $k \leq 4 s_n r$ makes $14 s_n r$ exceed the required threshold. At a large-radius saddle, the source first-moment estimate is $k \leq 6N^2$, for N in (4.14). Thus $k \geq 6(12 \cdot 10^6)^2$ forces $N \geq 12 \cdot 10^6$. Apply Proposition 4.2 and (4.3). □

Thus every possible failure is confined to a fixed width from an endpoint. This localization does not prove the remaining edge inequalities. A separate source-specific argument improves the proved edge range to

$$4 \min(k, d_n - k) \leq s_n, \qquad 1 \leq k, \quad k + 1 \leq d_n. \tag{5.1}$$

Here the conclusion is non-strict log-concavity. Its proof uses the exact dyadic logarithmic-derivative kernel and a one-sided bound on the normalized-ratio increments. It is distinct from the strict factor-28 estimate used in Theorem 1.2. Exact source certificates also prove that for $1 \leq n \leq 32$, D_n is log-concave exactly when $n \notin \{3, 5, 6, 7\}$. These results leave the full Conjecture 4 open.

5.2. **Binary log-concavity.** The revised Conjecture 7 asserts log-concavity of N_n for every $n \geq 6$. In addition to Theorem 1.1, the formal development proves the following two adjacent-center inequalities.

Theorem 5.2. *For every $n \geq 6$, with $c = c_n$,*

$$N_n[c-2] N_n[c] \leq N_n[c-1]^2, \qquad N_n[c] N_n[c+2] \leq N_n[c+1]^2. \tag{5.2}$$

Proof. For even $n \geq 100$, the source recurrence gives nonnegative additive curvature $2N_n[c-1] - N_n[c-2] - N_n[c]$. For the smoothing summand this follows by extracting two factors of $1+x$: their coefficient convolution turns symmetric unimodality into additive concavity immediately below the new center. The geometric summand has the corresponding central-curvature bound from its source shape estimates. Add the bounds, and use $(u+w)^2 \geq 4uw$ to pass from additive concavity to log-concavity. Reflection proves the second inequality. Exact source lists cover the even sizes 6 through 98, and (3.1) covers the odd sizes. □

The first-edge inequality $N_n[0]N_n[2] \leq N_n[1]^2$ is also proved for every n. These are partial coefficient ranges. Neither they nor unimodality establish the full system of quadratic inequalities in revised Conjecture 7.

Appendix A. Lean statements

A.1. **Public statements.** The main entry point is `BBFM/Results.lean`. Its public declarations include

Result	Lean declaration
Theorem 1.1	`BBFM.revisedConjecture6`
Theorem 1.2, non-strict form	`BBFM.conjecture4_large_n`
Strict internal form	`BBFM.conjecture4_large_n_strict`
Classification through 32	`BBFM.conjecture4_through_32`
Theorem 5.2	`BBFM.binaryLogConcavityBesideCenter`

The source definitions are `numB` over $\mathbb{Z}[x]$ and `denReduced` over $\mathbb{Q}[x]$. The propositions `BBFM.RevisedConjecture7` and `BBFM.Conjecture4` are definitions of the remaining open targets, not assumed proofs.

A.2. **Locations of the proof components.** The following table gives paths relative to the repository root.

Component	Source module
Exact binary recurrence	`BBFM/Binary/Recurrence.lean`
Residual and amplitude	`BBFM/Binary/Residual.lean`
Triangle subtraction	`BBFM/Binary/Unimodality/TriangleSubtraction.lean`
Central-slope transfer	`BBFM/Binary/CentralSlope.lean`
Invariant step	`BBFM/Binary/ShapeAssembly.lean`
Finite bases and induction	`BBFM/Binary/Unimodality/FiniteBases.lean`; `BBFM/Binary/AllUnimodal.lean`
Exact denominator product	`BBFM/Denominator/Bridge/Identification.lean`
Sharp moment bounds	`BBFM/Denominator/BernoulliSmallMoment.lean` and its imports
Fourier criterion	`BBFM/Denominator/MomentCriterion.lean`
Saddle discharges	`BBFM/Denominator/MomentSaddles.lean`
Strict coefficient edge	`BBFM/Denominator/EdgeTwentyEight.lean`
Eventual source theorem	`BBFM/Denominator/MomentEventualSource.lean`
Uniform source interior	`BBFM/Denominator/MomentSourceInterior.lean`
Reflected factor-4 edge	`BBFM/Denominator/SourceFourEdges.lean`
Adjacent-center binary LC	`BBFM/Binary/SourceCentralLCFinite.lean`

A.3. **Attribution.** BBFM formulated the conjectures and proved the structural product formula. The binary definitions and the denominator root-multiplicity excerpt are credited to the pinned Axiom repository [3]. Source references and licensing information appear in `docs/ATTRIBUTION.md`.

References

[1] C. Ballantine, G. Beck, B. Feigon and K. Maurischat, *Reciprocals of subsum polynomials*, revised version, arXiv:2605.10512v2, 2026. `https://arxiv.org/html/2605.10512v2`.

[2] E. Chen, K. Ono and J. Zhang, *Reciprocals of partition polynomials*, arXiv:2605.21718v2, 2026. `https://arxiv.org/html/2605.21718v2`.

[3] Axiom Math, *PartitionPolynomial*, Lean source repository, revision `b2a9e8d`. `https://github.com/AxiomMath/PartitionPolynomial`. Definition files: `Conjecture5/solution.lean` and `Conjecture2/solution.lean`, both under `PartitionPolynomial/`.

[4] *BBFM conjectures on subsum polynomials*, accompanying Lean 4 repository `bbfm-conjectures`, 2026. `https://github.com/anish-lakkapragada/bbfm-conjectures`. Public statements: `BBFM/Results.lean`.